

A tight stability version of the Complete Intersection Theorem

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July 12, 2016

Abstract

A set family \mathcal{F} is said to be *t-intersecting* if any two sets in \mathcal{F} share at least t elements. The Complete Intersection Theorem of Ahlswede and Khachatrian (1997) determines the maximal size $f(n, k, t)$ of a t -intersecting family of k -element subsets of $\{1, 2, \dots, n\}$, and gives a full characterisation of the extremal families.

In this paper, we prove the following ‘stability version’ of the theorem: if k/n is bounded away from 0 and $1/2$, and \mathcal{F} is a t -intersecting family of k -element subsets of $\{1, 2, \dots, n\}$ such that $|\mathcal{F}| \geq f(n, k, t) - O\binom{n-d}{k}$, then there exists an extremal family \mathcal{G} such that $|\mathcal{F} \setminus \mathcal{G}| = O\binom{n-d}{k-d}$. For fixed t , this assertion is tight up to a constant factor. This proves a conjecture of Friedgut from 2008.

Our proof combines classical shifting arguments with a ‘bootstrapping’ method based upon an isoperimetric argument.

1 Introduction

We write $[n] := \{1, 2, \dots, n\}$, and $[n]^{(k)} := \{A \subset [n] : |A| = k\}$. A family $\mathcal{F} \subset \mathcal{P}([n])$ (i.e., a family of subsets of $[n]$) is said to be *increasing* if $A \supset B \in \mathcal{F}$ implies $A \in \mathcal{F}$, and *intersecting* if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$. For $t \in \mathbb{N}$, \mathcal{F} is said to be *t-intersecting* if for any $A, B \in \mathcal{F}$, $|A \cap B| \geq t$. A *dictatorship* is a family of the form $\{S | i \in S\} := \mathcal{D}_i$ for some $i \in [n]$, and a *t-umvirate* is a family of the form $\{S | B \subset S\} := \mathcal{S}_B$, for some $B \in [n]^{(t)}$.

The classical Erdős-Ko-Rado theorem [11] determines the maximal size of an intersecting family $\mathcal{F} \subset [n]^{(k)}$.

Theorem 1.1 (Erdős, Ko, and Rado, 1961). *Let $k < n/2$, and let $\mathcal{F} \subset [n]^{(k)}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Equality holds if and only if \mathcal{F} is a dictatorship.*

In the same paper [11], Erdős, Ko and Rado showed that for n sufficiently large depending on k and t , the maximal size of a t -intersecting $\mathcal{F} \subset [n]^{(k)}$ is $\binom{n-t}{k-t}$. For general $n, k, t \in \mathbb{N}$, we write $f(n, k, t)$ for this maximum. Determination of $f(n, k, t)$ for general $n, k, t \in \mathbb{N}$ remained a major open problem for more than three decades, until it was accomplished by Ahlswede and Khachatrian [2] in their *Complete Intersection Theorem*:

Theorem 1.2 (Ahlswede and Khachatrian, 1997). *For any $n, k, t, r \in \mathbb{N}$, let $\mathcal{F}_{n,k,t,r} = \{S \in [n]^{(k)} : |S \cap [t+2r]| \geq t+r\}$. Let $\mathcal{F} \subset [n]^{(k)}$ be a t -intersecting family. Then $|\mathcal{F}| \leq \max_r |\mathcal{F}_{n,k,t,r}|$, and if equality holds then \mathcal{F} is isomorphic to one of the $\mathcal{F}_{n,k,t,r}$ families. In particular, if $n \geq (k-t+1)(t+1)$, then $|\mathcal{F}| \leq |\mathcal{F}_{n,k,t,0}| = \binom{n-t}{k-t}$.*

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(We say $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$ are *isomorphic* if there exists a permutation $\sigma \in \text{Sym}([n])$ such that $\mathcal{G} = \{\sigma(S) : S \in \mathcal{F}\}$.)

Along the years, numerous works have studied *stability versions* of the Erdős-Ko-Rado (EKR) and the Ahlswede-Khachatrian (AK) theorems, asserting that if the size of a family is *close to the maximum possible*, then that family is *close* (in an appropriate sense) to an extremal family.

The first to study such questions were Hilton and Milner [17], who showed in 1967 that if the size of an intersecting family is *very* close to $\binom{n-1}{k-1}$, then the family is *contained* in a dictatorship. A similar result for the complete intersection theorem in the domain $n \geq (k-t+1)(t+1)$ was obtained in 1996 by Ahlswede and Khachatrian [1]. A simpler proof of the latter result was presented by Balogh and Mubayi [5], and an alternative result of the same class was obtained by Anstee and Keevash [3].

For families whose size is not very close to the maximum, Frankl [13] obtained in 1987 a strong stability version of the EKR theorem which asserts that if an intersecting family \mathcal{F} satisfies $|\mathcal{F}| \geq (1-\epsilon)\binom{n-1}{k-1}$, then there exists a dictatorship \mathcal{D} such that $|\mathcal{F} \setminus \mathcal{D}| = O(\epsilon^{\log_{1-p} p})\binom{n}{k}$, where $p \approx k/n$. Frankl's result is tight and holds not only for $|\mathcal{F}|$ close to $\binom{n-1}{k-1}$ but rather for any $|\mathcal{F}| \geq c\binom{n-2}{k-2}$. Proofs of somewhat weaker results using entirely different techniques were later presented by Dinur and Friedgut [8], Friedgut [16] and Keevash [20]. In [21], Keevash and Mubayi used Frankl's result to prove an EKR-type theorem on set systems that do not contain a simplex or a cluster. Recently, a different notion of stability for the EKR theorem was suggested by Bollobás et al. [6] and already studied in several subsequent papers (e.g., [4, 7]).

The case of the AK theorem appeared much harder. The first stability result was obtained by Friedgut [16], who showed in 2008 that for any $\epsilon \geq \sqrt{(\log n)/n}$, $\zeta > 0$ and $\zeta n < k < (1/(t+1) - \zeta)n$, if a t -intersecting $\mathcal{F} \subset [n]^{(k)}$ satisfies $|\mathcal{F}| \geq f(n, k, t)(1-\epsilon)$, then there exists a t -umvirate \mathcal{G} such that $|\mathcal{F} \setminus \mathcal{G}| = O_{t,\zeta}(\epsilon)\binom{n}{k}$. The proof of Friedgut uses Fourier analysis and spectral methods. Recently, Ellis et al. [10] proved a strong version of Friedgut's result, which asserts that under the conditions of Friedgut's theorem, $|\mathcal{F} \setminus \mathcal{G}| = O(\epsilon^{\log_{1-p} p})\binom{n}{k}$ (where $p \approx k/n$) for some t -umvirate \mathcal{G} , and showed that it is tight by an explicit example. The main technique of [10] is to utilize isoperimetric inequalities on the hypercube.

All the results described above apply only in the so-called 'principal domain' $k < n/(t+1)$, in which the extremal example has the simple structure of a t -umvirate. In the general case, where the extremal examples are the more complex families $\mathcal{F}_{n,k,t,r}$, no stability result has been obtained so far (to the best of our knowledge). This situation resembles the history of the 'exact' results, where Theorem 1.2 was proved for $k < n/(t+1)$ already by Wilson [25] in 1984, and it took 13 more years until the general case was resolved by Ahlswede and Khachatrian.

The main conjecture stated in Friedgut's 2008 paper [16] is that his stability result holds for all $\zeta n < k < (1/2 - \zeta)n$. To state the conjecture, we need an additional explanation.

A direct computation shows that for any $\beta \in (0, 1/2)$ and any $t \in \mathbb{N}$, there is either a unique value of r or two consecutive values of r that asymptotically maximize $|\mathcal{F}_{n, \lfloor \beta n \rfloor, t, r}|$ (as $n \rightarrow \infty$). We say that β is *non-singular* for t if there is a unique such value of r , which we then denote by $r^* = r^*(\beta, t)$. Otherwise we say that β is *singular* for t , and we let r^* and $r^* + 1$ be the two extremal values of r .

Conjecture 1.3. [16, Conjecture 4.1] *Let $t \in \mathbb{N}$, let $\zeta > 0$, let $\beta \in [\zeta, 1/2 - \zeta]$ be non-singular for t , let $\epsilon > 0$, and let $k = \lfloor \beta n \rfloor$. If $\mathcal{F} \subset [n]^{(k)}$ is a t -intersecting family such that $|\mathcal{F}| \geq (1-\epsilon)|\mathcal{F}(n, k, t, r^*)|$, then there exists a set $B \subset [n]$ of size $t + 2r^*$ such that*

$|\{A \in \mathcal{F} : |A \cap B| \geq t + r^*\}| \geq (1 - O_{t,\zeta}(\epsilon))|\mathcal{F}|$. If β is singular for t , then either the above holds or the corresponding statement for $r^* + 1$ holds.

In this paper, we prove Conjecture 1.3, establishing a stability version of the Ahlswede-Khachatrian theorem for all $k < (1/2 - \zeta)n$. Moreover, we show that the assertion can be strengthened to $|\{A \in \mathcal{F} : |A \cap B| \geq t + r^*\}| \geq (1 - O_{t,\zeta}(\epsilon^{\log_{1-\beta}\beta}))|\mathcal{F}|$ (or $|\{A \in \mathcal{F} : |A \cap B| \geq t + r^* + 1\}| \geq (1 - O_{t,\zeta}(\epsilon^{\log_{1-\beta}\beta}))|\mathcal{F}|$), if β is singular for t), and that this is tight, up to the dependence on t and ζ .

Before stating our main result, let us elaborate on the techniques we use. As in previous works in this direction (e.g., [8, 10, 16]), it is more convenient for us to work with the *biased measure* version of the EKR and AK theorems. In this version, we consider (t) -intersecting families $\mathcal{F} \subset \mathcal{P}([n])$, and seek to maximize their biased measure $\mu_p(\mathcal{F})$, defined by $\mu_p(\mathcal{F}) := \sum_{S \in \mathcal{F}} p^{|S|}(1-p)^{n-|S|}$. The biased version of the AK theorem (presented clearly in [12]), is as follows.

Theorem 1.4 (Biased AK Theorem). *Let $t \in \mathbb{N}$, let $0 < p < 1/2$, and let $\mathcal{F} \subset \mathcal{P}([n])$ be a t -intersecting family. Then $\mu_p(\mathcal{F}) \leq f(n, p, t) := \max_r \mu_p(\mathcal{F}_{n,t,r})$, where $\mathcal{F}_{n,t,r} := \{S \subset [n] : |S \cap [t + 2r]| \geq t + r\}$, and equality holds iff \mathcal{F} is isomorphic to one of the $\mathcal{F}_{n,t,r}$ families. In particular, if $p < 1/(t + 1)$, then $\mu_p(\mathcal{F}) \leq \mu_p(\tilde{\mathcal{F}}_{n,t,0}) = p^t$, with equality iff $\mathcal{F} \cong \mathcal{F}_{n,t,0}$.*

We prove the following stability version of Theorem 1.4.

Theorem 1.5. *For any $t \in \mathbb{N}$ and any $\zeta > 0$, there exists $C = C(t, \zeta) > 0$ such that the following holds. Let $p \in [\zeta, \frac{1}{2} - \zeta]$, and let $\epsilon > 0$. If $\mathcal{F} \subset \mathcal{P}([n])$ is a t -intersecting family such that $\mu_p(\mathcal{F}) \geq f(n, p, t)(1 - \epsilon)$, then there exists a family \mathcal{G} isomorphic to some $\mathcal{F}_{n,t,r}$, such that $\mu_p(\mathcal{F} \setminus \mathcal{G}) \leq C\epsilon^{\log_{1-p}p}$.*

In particular, if $\frac{r^}{t+2r^*-1} + \zeta < p < \frac{r^*+1}{t+2r^*+1} - \zeta$ for some $r^* \in \mathbb{N}$, then the above holds with $\mathcal{G} \cong \mathcal{F}_{n,t,r^*}$.*

Theorem 1.5 is tight, up to a factor depending only on t and ζ , for the families $\mathcal{G}_{t,r,s}$, defined by:

$$\begin{aligned} \mathcal{G}_{t,r,s} = & \{A \subset \mathcal{P}([n]) : |A \cap [t + 2r]| \geq t + r, A \cap \{t + 2r + 1, \dots, t + 2r + s\} \neq \emptyset\} \\ & \cup \{A \subset \mathcal{P}([n]) : |A \cap [t + 2r]| = t + r - 1, \{t + 2r + 1, \dots, t + 2r + s\} \subset A\}, \end{aligned}$$

for all sufficiently large s . The computation showing this is presented in Section 4.1.

Theorem 1.5 follows from combination of three ingredients:

- A **bootstrapping lemma** showing that if a t -intersecting \mathcal{F} is *somewhat close* to some $\mathcal{F}_{n,t,r}$ then it must be *very close* to that $\mathcal{F}_{n,t,r}$. More precisely, there exists $c > 0$ such that if $\mu_p(\mathcal{F} \setminus \mathcal{F}_{n,t,r}) > c$, and if \mathcal{G} is another t -intersecting family which is a *small modification* of \mathcal{F} , in the sense that $\mu_p(\mathcal{F} \setminus \mathcal{G}) < c/2$, then $\mu_p(\mathcal{G} \setminus \mathcal{F}_{n,t,r}) > c$. Hence, there is a ‘barrier’ which one cannot cross while making only small modifications.
- A **shifting argument** showing that given a t -intersecting family \mathcal{F} , one can transform it into a *junta* (i.e., a function that depends on only $O(1)$ coordinates) $\tilde{\mathcal{F}}$ with $\mu_p(\tilde{\mathcal{F}}) \geq \mu_p(\mathcal{F})$ by a series of small modifications.
- An observation that if a t -intersecting junta $\tilde{\mathcal{F}}$ satisfies $\mu_p(\tilde{\mathcal{F}}) > f(n, p, t)(1 - \epsilon)$ for a sufficiently small ϵ , then it must be isomorphic to one of the $\mathcal{F}_{n,t,r}$ ’s.

While the shifting part is based on ‘classical’ shifting arguments summarized in [12], the bootstrapping relies on a recently introduced isoperimetric argument [10]. It seems that the combination of the classical shifting tools with isoperimetry is the main novelty of the proof.

With Theorem 1.5 in hand, we turn to prove our main result.

Theorem 1.6. *Let $n, t, d \in \mathbb{N}$, $\zeta \in (0, 1/2)$, and $k \in (\zeta n, (\frac{1}{2} - \zeta)n)$. There exists $C = C(t, \zeta) > 0$ such the following holds. Let $\mathcal{F} \subset [n]^{(k)}$ be a t -intersecting family with $|\mathcal{F}| > f(n, k, t) - \frac{1}{C} \binom{n-d}{k}$. Then there exists \mathcal{G} isomorphic to some $\mathcal{F}_{n, k, t, r}$ such that $|\mathcal{F} \setminus \mathcal{G}| < C \binom{n-d}{k-d}$.*

Theorem 1.6 is tight (up to a factor depending only on t and ζ) for the families $\mathcal{G}_{t, r, s} \cap [n]^{(k)}$, where $\mathcal{G}_{t, r, s}$ are the tightness examples for Theorem 1.5, defined above. It is easy to see that Theorem 1.6 implies Conjecture 1.3, in the stronger form above with $\epsilon^{\log_{1-\beta} \beta}$ in place of ϵ . Combining it with the stability version of the AK theorem for $k < n/(t+1)$ proved in [10], one obtains a tight stability version of the complete intersection theorem for all $k < (1/2 - \zeta)n$.

The derivation of Theorem 1.6 from Theorem 1.5 uses another bootstrapping argument. First we use a standard reduction from the setting of k -element subsets of $[n]$ to the biased setting to obtain a weak stability theorem. Then, we bootstrap the result using a recently introduced argument [10] which relies on the Kruskal-Katona theorem.

This paper is organized as follows. In Section 2 we present previous results which we use in our proofs – concerning juntas, influences, shifting techniques, cross-intersecting families, and reduction methods. In Section 3 we prove the bootstrapping lemma, and in Section 4 we prove Theorem 1.5. Finally, the proof of Theorem 1.6 is presented in Section 5.

2 Techniques and previous results we use

Our results partially rely on several previous results in extremal and probabilistic combinatorics. In this section we present notations, definitions, and previous results and techniques that will be used in the sequel. As some of the results were not proved in the form we use them, we present their proofs for sake of completeness. The reader might find it useful for him to flip through this section and go back to the specific results when they are used in the sequel.

2.1 Notations

Throughout Sections 2–4, \mathcal{F} denotes a set family, i.e., a subset of $\mathcal{P}([n])$. The k ’th layer of \mathcal{F} is $\mathcal{F} \cap [n]^{(k)} =: \mathcal{F}^{(k)}$. We (almost) always assume that $0 < p \leq 1/2 - \zeta$, for a fixed $\zeta > 0$.

$\mathbb{AK}_{t, \zeta}^n$ denotes the collection of extremal families for the biased AK theorem corresponding to (n, p, t) (i.e., the collection of all t -intersecting \mathcal{F} with $\mu_p(\mathcal{F}) = f(n, p, t)$ for some $0 < p \leq 1/2 - \zeta$). Note that for any $\mathcal{A} \in \mathbb{AK}_{t, \zeta}^n$ and any $p \in [\zeta, 1/2 - \zeta]$, we have $\mu_p(\mathcal{A}) = \Theta_{t, \zeta}(1)$.

For any \mathcal{F} , we denote by $\mu_p(\mathcal{F} \setminus \mathbb{AK}_{t, \zeta}^n)$ the ‘minimal distance’ $\min_{\mathcal{G} \in \mathbb{AK}_{t, \zeta}^n} \mu_p(\mathcal{F} \setminus \mathcal{G})$.

For $k \in [n]$, $(\mathbb{AK}_{t, \zeta}^n)^{(k)}$ denotes the k ’th layers of elements of $\mathbb{AK}_{t, \zeta}^n$, i.e., $\{\mathcal{A} \cap [n]^{(k)} : \mathcal{A} \in \mathbb{AK}_{t, \zeta}^n\}$. For $\mathcal{F} \subset [n]^{(k)}$, we denote by $|\mathcal{F} \setminus (\mathbb{AK}_{t, \zeta}^n)^{(k)}|$ the ‘minimal distance’ $\min_{\mathcal{G} \in (\mathbb{AK}_{t, \zeta}^n)^{(k)}} |\mathcal{F} \setminus \mathcal{G}|$.

For $E \subset D \subset [n]$, we denote $\mathcal{F}_D^E = \{A \subset [n] \setminus D : A \cup E \in \mathcal{F}\}$.

For a fixed $c > 0$, we say that \mathcal{G} is a c -small modification of \mathcal{F} if $\mu_p(\mathcal{F} \setminus \mathcal{G}) \leq c$.

We use the (now standard) ‘asymptotic notation’ as follows. If $f = f(x)$ and $g = g(x)$ are non-negative functions, we write $f = O(g)$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$ for all x . We write $f = \Omega(g)$ if there exists $c > 0$ such that $f(x) \geq cg(x)$ for all x , and we write $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. If $f = f(x; \alpha)$ and $g = g(x; \alpha)$ are non-negative functions, then we write $f = O_\alpha(g)$ if for all α , there exists $C = C(\alpha) > 0$ such that $f(x; \alpha) \leq Cg(x; \alpha)$ for all x . Similarly, we use the notation $f = \Omega_\alpha(g)$ and $f = \Theta_\alpha(g)$. (Here, we view α as a parameter; note that α may be vector-valued.)

2.2 Influences and Juntas

Definition 2.1. Let $\mathcal{F} \subset \mathcal{P}([n])$ and $i \in [n]$. The set of i -influential elements with respect to \mathcal{F} is

$$\mathcal{I}_i(\mathcal{F}) := \{S : |\{S, S \triangle \{i\}\} \cap \mathcal{F}| = 1\}.$$

The influence of the i 'th coordinate on \mathcal{F} with respect to μ_p is $I_i(\mathcal{F}) := \mu_p(\mathcal{I}_i(\mathcal{F}))$. The total influence of \mathcal{F} is $I(\mathcal{F}) := \sum_{i=1}^n I_i(\mathcal{F})$.

\mathcal{F} is said to be a j -junta if it depends on only j coordinates (i.e., if $I_i(\mathcal{F}) = 0$ for at least $n - j$ of the i 's).

Influences play an important role in a variety of applications in combinatorics, theoretical computer science, mathematical physics, social choice theory, etc. (see the survey [18]). A classical theorem of Friedgut [15] shows that a function whose total influence is small can be approximated by a junta. We shall use another relation between influences and juntas: if all influences of an increasing family are large, then the family must be a junta.

Proposition 2.2. Let $\mathcal{F} \subset \mathcal{P}([n])$ be increasing, let $0 < p < 1$ and let $c > 0$. If $\min_i (I_i(\mathcal{F})) \geq c$, then $n \leq \frac{1}{c^2 p(1-p)}$.

Proof. Let $f = \sum_{S \subset [n]} \hat{f}(S) \chi_S$ be the Fourier expansion of the characteristic function $f = 1_{\mathcal{F}}$, with respect to the measure μ_p . An easy calculation presented in [24, Proposition 8.45] shows that

$$I[\mathcal{F}] = \sum_{i=1}^n \frac{\hat{f}(\{i\})^2}{\sqrt{p(1-p)}}.$$

By the Cauchy-Schwarz inequality and Parseval's identity,

$$I[\mathcal{F}] = \sum_{i=1}^n \frac{\hat{f}(\{i\})^2}{\sqrt{p(1-p)}} \leq \frac{\sqrt{n} \sqrt{\sum_{i=1}^n \hat{f}(\{i\})^2}}{\sqrt{p(1-p)}} \leq \sqrt{\frac{n}{p(1-p)}} \|f\|_2 \leq \sqrt{\frac{n}{p(1-p)}}.$$

If $\min_i (I_i(\mathcal{F})) \geq c$, we have $cn \leq I(\mathcal{F}) \leq \sqrt{\frac{n}{p(1-p)}}$. Rearranging yields the assertion. \square

Another observation we use is that the measure of a t -intersecting junta \mathcal{F} cannot be ‘too close’ to $f(n, p, t)$, unless \mathcal{F} is isomorphic to one of the $\mathcal{F}_{n,t,r}$ families (which are, of course, juntas).

Proposition 2.3. Let $t \in \mathbb{N}$, let $\zeta > 0$, and let $m \in \mathbb{N}$. There exists $\epsilon = \epsilon(t, \zeta, m) > 0$ such that if $0 < p \leq 1/2 - \zeta$, and $\mathcal{F} \subset \mathcal{P}([m])$ is a t -intersecting family satisfying $\mu_p(\mathcal{F}) \geq f(m, p, t)(1 - \epsilon)$, then $\mathcal{F} \in \mathbb{A}_{p,t}^m$.

Proof. By Theorem 1.4, for any $r \in \mathbb{N} \cup \{0\}$ and any $\frac{r}{t+2r-1} \leq p \leq \frac{r+1}{t+2r+1}$, any t -intersecting $\mathcal{F} \in \mathcal{P}([m]) \setminus \mathbb{AK}_{t,\zeta}^m$ satisfies $\mu_p(\mathcal{F}) < \mu_p(\mathcal{F}_{t,r}^m)$. Consider the set

$$C_r^m = \{\mu_q(\mathcal{F}_{m,t,r}) - \mu_q(\mathcal{G}) : \mathcal{G} \in \mathcal{P}([m]) \setminus \mathbb{AK}_{q,t}^m, \mathcal{G} \text{ is } t\text{-intersecting}, \\ \frac{r}{t+2r-1} \leq q \leq \frac{r+1}{t+2r+1}\}.$$

As m is fixed, C_r^m is compact and all its elements are positive. Hence, $c_r^m = \min(C_r^m) > 0$. We have

$$[0, 1/2 - \zeta] \subset \bigcup_{r=0}^{\ell} \left[\frac{r}{t+2r-1}, \frac{r+1}{t+2r+1} \right],$$

for some $\ell = \ell(t, \zeta) \in \mathbb{N}$. Let $\epsilon = \min_{0 \leq r \leq \ell} c_r^m$. It is clear by the choice of ϵ that for all $p \in (0, 1/2 - \zeta]$, if $\mathcal{F} \subset \mathcal{P}([m])$ is a t -intersecting family with $\mu_p(\mathcal{F}) \geq f(m, p, t)(1 - \epsilon) > f(m, p, t) - \epsilon$ then we must have $\mathcal{F} \in \mathbb{AK}_{p,t}^m$. This completes the proof. \square

2.3 Shifting

Shifting (a.k.a. ‘compression’) is one of the most classical techniques in extremal combinatorics (see, e.g., [14]).

Definition 2.4. For $\mathcal{F} \subset \mathcal{P}([n])$, a set $A \in \mathcal{F}$, and $1 \leq j < i \leq n$, the shifting operator \mathcal{S}_{ij} is defined as follows: $\mathcal{S}_{ij}(A) = A \setminus \{i\} \cup \{j\}$ if $i \in A$, $j \notin A$, and $A \setminus \{i\} \cup \{j\} \notin \mathcal{F}$; and $\mathcal{S}_{ij}(A) = A$ otherwise. We define $\mathcal{S}_{ij}(\mathcal{F}) = \{\mathcal{S}_{ij}(A) : A \in \mathcal{F}\}$.

\mathcal{F} is called n -compressed if $A \setminus \{n\} \cup \{j\} \in \mathcal{F}$ for all $A \in \mathcal{F}$ such that $A \cap \{j, n\} = n$, i.e., if $\mathcal{S}_{nj}(\mathcal{F}) = \mathcal{F}$ for all $j < n$. \mathcal{F} is called shifted if $\mathcal{S}_{ij}(\mathcal{F}) = \mathcal{F}$ for all $j < i$.

The following properties of the shifting operator are easy to check.

Claim 2.5. Let $\mathcal{F} \subset \mathcal{P}([n])$ be increasing and t -intersecting. Then $\mathcal{S}_{ij}(\mathcal{F})$ satisfies the following properties:

- (a) $\mu_p(\mathcal{S}_{ij}(\mathcal{F})) = \mu_p(\mathcal{F})$,
- (b) $\mu_p(\mathcal{F} \setminus \mathcal{S}_{ij}(\mathcal{F})) \leq I_i(\mathcal{F})$,
- (c) $I_i(\mathcal{S}_{ij}(\mathcal{F})) \leq I_i[\mathcal{F}]$, with equality if and only if $\mathcal{S}_{ij}(\mathcal{F}) = \mathcal{F}$,
- (d) $\mathcal{S}_{ij}(\mathcal{F})$ is increasing and t -intersecting.

2.3.1 n -compression by small modifications

The following proposition shows that any increasing t -intersecting \mathcal{F} can be transformed into an n -compressed increasing t -intersecting \mathcal{G} with $\mu_p(\mathcal{G}) = \mu_p(\mathcal{F})$ by a sequence of c -small modifications, where $c = I_n(\mathcal{F})$.

Proposition 2.6. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing t -intersecting family that is not n -compressed. Denote $\delta = I_n(\mathcal{F})$. Then there exist families $\mathcal{F} = \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m \subset \mathcal{P}([n])$, such that \mathcal{F}_m is n -compressed and for each $i \in [m]$ we have:

- (a) $\mu_p(\mathcal{F}_i) = \mu_p(\mathcal{F})$,
- (b) \mathcal{F}_i is increasing and t -intersecting,
- (c) $\mu_p(\mathcal{F}_{i-1} \setminus \mathcal{F}_i) \leq \delta$,
- (d) $I_n(\mathcal{F}_i) < \delta$.

Proof. We define \mathcal{F}_i inductively. Suppose that \mathcal{F}_{i-1} is not n -compressed. Then for some $j \in [n]$, we have $\mathcal{S}_{nj}(\mathcal{F}_{i-1}) \neq \mathcal{F}_{i-1}$. We choose such a j arbitrarily and define $\mathcal{F}_i = \mathcal{S}_{nj}(\mathcal{F}_{i-1})$. By Claim 2.5, \mathcal{F}_i satisfies the desired properties. Thus, we only need to show that for some $m \in \mathbb{N}$, \mathcal{F}_m is n -compressed. Indeed, by Claim 2.5, $I_n(\mathcal{F}_i)$ is strictly decreasing (as a function of i). Since all of $\{I_n(\mathcal{F}_i) : i \in \mathbb{N}\}$ belong to a finite set of values, this process cannot last forever. \square

2.3.2 Increasing the measure by a small modification

Our next goal is to show that if \mathcal{F} is an n -compressed, increasing, t -intersecting family, then it can be transformed to a t -intersecting \mathcal{G} with $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$ by a c -small modification, where $c = I_n(\mathcal{F})$. That is, if \mathcal{F} is already n -compressed, then its measure can be *increased* by a small modification, without sacrificing the t -intersection property.

To show this, we need several claims. These claims were proved in [12] under the assumption that \mathcal{F} is shifted, but it turns out that the proof of [12] applies also under the weaker assumption that \mathcal{F} is n -compressed. For sake of completeness, we present the claims below.

Lemma 2.7. *Let $\mathcal{F} \subset \mathcal{P}([n])$ be a t -intersecting n -compressed family and let $a, b \in [n]$. Let $A \in \mathcal{F}^{(a)}$ and $B \in \mathcal{F}^{(b)}$ be such that $|A \cap B| = t$ and $n \in A \cap B$. Then $a + b = n + t$, and $A \cup B = [n]$.*

Proof. It is clearly sufficient to show that $A \cup B = [n]$. Suppose for a contradiction that $i \notin A \cup B$. As \mathcal{F} is n -compressed, we have $A' = A \setminus \{n\} \cup \{i\} \in \mathcal{F}$. However, $|A' \cap B| = t - 1$, contradicting the fact that \mathcal{F} is t -intersecting. \square

The next proposition shows that if an n -compressed t -intersecting family \mathcal{F} satisfies $\mathcal{F} \cap \mathcal{I}_n(\mathcal{F}) \not\subseteq [n]^{\binom{n+t}{2}}$, then the measure of \mathcal{F} can be increased by a small modification.

Proposition 2.8. *Let $0 < p < 1/2$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing t -intersecting n -compressed family. Denote $\mathcal{I}_n = \mathcal{I}_n(\mathcal{F})$. Let $a \neq b \in [n]$ be such that $a + b = n + t$. Then the families*

$$\mathcal{G}_1 := (\mathcal{F} \setminus (\mathcal{I}_n \cap \mathcal{F})^{(a)}) \cup (\mathcal{I}_n \setminus \mathcal{F})^{(b-1)} \quad \text{and} \quad \mathcal{G}_2 := (\mathcal{F} \setminus (\mathcal{I}_n \cap \mathcal{F})^{(b)}) \cup (\mathcal{I}_n \setminus \mathcal{F})^{(a-1)}$$

are t -intersecting, and

$$\mu_p(\mathcal{F}) \leq \max\{\mu_p(\mathcal{G}_1), \mu_p(\mathcal{G}_2)\}, \tag{1}$$

with equality only if $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{F}$.

Proof. W.l.o.g. we show that \mathcal{G}_1 is t -intersecting. Let $A, B \in \mathcal{G}_1$, and suppose for a contradiction that $|A \cap B| \leq t - 1$. Hence, we have either $A \in (\mathcal{I}_n \setminus \mathcal{F})^{(b-1)}$ or $B \in (\mathcal{I}_n \setminus \mathcal{F})^{(b-1)}$ or both. Assume, w.l.o.g., $B \in (\mathcal{I}_n \setminus \mathcal{F})^{(b-1)}$. Then $B' := B \cup \{n\} \in (\mathcal{I}_n \cap \mathcal{F})^{(b)}$. Note that we have $A' := A \cup \{n\} \in \mathcal{F}$. Indeed, either $A \in \mathcal{F}$ and then $A' \in \mathcal{F}$ since \mathcal{F} is increasing, or $A \in (\mathcal{I}_n \setminus \mathcal{F})^{(b-1)}$ and then $A' \in (\mathcal{I}_n \cap \mathcal{F})^{(b)}$. Since \mathcal{F} is t -intersecting, this implies

$$t \leq |A' \cap B'| \leq |A \cap B| + 1 \leq t. \tag{2}$$

This allows applying Lemma 2.7 to A', B' , to get $|A'| = a$.

Now, as $|(A' \setminus \{n\}) \cap B'| = t - 1$ and \mathcal{F} is t -intersecting, we have $A' \setminus \{n\} \notin \mathcal{F}$. Hence, $A' \in (\mathcal{I}_n \cap \mathcal{F})^{(a)}$, which yields $A' \notin \mathcal{G}_1$. As $A \in \mathcal{G}_1$, we must have $A = A' \setminus \{n\} \notin \mathcal{F}$. By the construction of \mathcal{G}_1 , this means that $A \in (\mathcal{I}_n \setminus \mathcal{F})^{(b-1)}$, and therefore, $|A'| = a = b$, a contradiction.

The proof of (1) is a straightforward calculation. Write $\mathcal{A}_1 = (\mathcal{I}_n \cap \mathcal{F})^{(a)}$ and $\mathcal{A}_2 = (\mathcal{I}_n \cap \mathcal{F})^{(b)}$. Suppose w.l.o.g. that $\mu_p(\mathcal{A}_1) \geq \mu_p(\mathcal{A}_2)$. Then

$$\mu_p(\mathcal{G}_2) = \mu_p(\mathcal{F}) - \mu_p(\mathcal{A}_2) + \frac{1-p}{p} \mu_p(\mathcal{A}_1) > \mu_p(\mathcal{F}),$$

as asserted. It is also clear that equality can hold only if $\mathcal{A}_1 = \mathcal{A}_2 = \emptyset$, that is, if $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{F}$. This completes the proof. \square

The following proposition complements the previous one by showing that the measure of an n -compressed t -intersecting \mathcal{F} can be increased by a small modification even if $\mathcal{F} \cap \mathcal{I}_n(\mathcal{F}) \subset [n]^{\binom{n+t}{2}}$. Recall that \mathcal{D}_i denotes a dictatorship $\mathcal{D}_i := \{A \in \mathcal{P}([n]) : i \in A\}$.

Proposition 2.9. *Let $n, t \in \mathbb{N}$ such that $n + t$ is even. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing n -compressed t -intersecting family such that $I_n(\mathcal{F}) > 0$. Denote $a = \frac{n+t}{2}$. For $1 \leq i \leq n-1$, let*

$$\mathcal{G}_i = (\mathcal{F} \setminus (\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i)^{(a)}) \cup (\mathcal{I}_n \setminus (\mathcal{F} \cup \mathcal{D}_i))^{(a-1)}.$$

Then the families \mathcal{G}_i are t -intersecting. Moreover, if $0 < p \leq 1/2 - \zeta$, $n > t/(2\zeta)$ and $(\mathcal{I}_n \cap \mathcal{F})^{(a)} \neq \emptyset$, then

$$\max_{i \in [n-1]} \{\mu_p(\mathcal{G}_i)\} > \mu_p(\mathcal{F}). \quad (3)$$

Proof. First we prove that for all i , \mathcal{G}_i is t -intersecting. Let $A, B \in \mathcal{G}_i$, and suppose for a contradiction that $|A \cap B| \leq t-1$. Denote $A' := A \cup \{n\}$ and $B' := B \cup \{n\}$, and assume w.l.o.g. $B \notin \mathcal{F}$, and hence, $|B'| = a$ and $i \notin B'$.

By the same argument as in the proof of Proposition 2.8, we have $A' \in \mathcal{F} \cap \mathcal{I}_n$. On the other hand, by Lemma 2.7 (applied for A', B') we have $A' \cup B' = [n]$, and hence, $i \in A'$ and $|A'| = a$. Thus, $A' \in (\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i)^{(a)}$, which implies $A' \notin \mathcal{G}_i$. This is a contradiction, as $A \in \mathcal{G}_i$ and \mathcal{G}_i is increasing.

Now we prove Equation (3). Note that for any $i \leq n-1$, there is a one-to-one correspondence between the families $(\mathcal{I}_n \setminus (\mathcal{F} \cup \mathcal{D}_i))^{(a-1)}$ and $(\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i^c)^{(a)}$. Hence,

$$\begin{aligned} \mu_p(\mathcal{G}_i) &= \mu_p(\mathcal{F}) - \mu_p\left((\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i)^{(a)}\right) + \left(\frac{1-p}{p}\right) \mu_p\left((\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i^c)^{(a)}\right) \\ &= \mu_p(\mathcal{F}) - \mu_p\left((\mathcal{F} \cap \mathcal{I}_n)^{(a)}\right) + \mu_p\left((\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i^c)^{(a)}\right) + \left(\frac{1-p}{p}\right) \mu_p\left((\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i^c)^{(a)}\right) \\ &= \mu_p(\mathcal{F}) - \mu_p\left((\mathcal{F} \cap \mathcal{I}_n)^{(a)}\right) + \frac{1}{p} \mu_p\left((\mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i^c)^{(a)}\right). \end{aligned}$$

Let $\mathcal{K}_i := \mathcal{F} \cap \mathcal{I}_n \cap \mathcal{D}_i^c$. We have

$$\mathbb{E}_i[\mu_p(\mathcal{K}_i)] = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{A \in \mathcal{K}_i} \mu_p(A) = \frac{1}{n-1} \sum_{A \in (\mathcal{F} \cap \mathcal{I}_n)^{(a)}} \sum_{\{i: i \notin A\}} \mu_p(A) = \frac{n-a}{n-1} \mu_p((\mathcal{F} \cap \mathcal{I}_n)^{(a)}).$$

Thus, there exists $i \in [n-1]$ such that $\mu_p(\mathcal{K}_i) \geq \frac{n-a}{n-1} \mu_p((\mathcal{F} \cap \mathcal{I}_n)^{(a)})$. This implies

$$\max \{\mu_p(\mathcal{G}_i)\} \geq \mu_p(\mathcal{F}) + \frac{n-a-(n-1)p}{(n-1)p} \mu_p\left((\mathcal{F} \cap \mathcal{I}_n)^{(a)}\right) > \mu_p(\mathcal{F}),$$

where the last inequality holds since $n-a-(n-1)p > 0$ for all $n > t/(2\zeta)$. This completes the proof. \square

Combining Propositions 2.8 and 2.9 we obtain:

Corollary 2.10. *Let $\zeta > 0$, let $0 < p \leq 1/2 - \zeta$ and let $n \in \mathbb{N}$ with $n > t/(2\zeta)$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing n -compressed t -intersecting family that depends on the n th coordinate (i.e., $I_n(\mathcal{F}) > 0$). Then there exists a t -intersecting family $\mathcal{G} \subset \mathcal{P}([n])$, such that $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$ and $\mu_p(\mathcal{F} \setminus \mathcal{G}) \leq I_n(\mathcal{F})$.*

While \mathcal{G} obtained in Corollary 2.10 is t -intersecting, it is not necessarily n -compressed. However, by Proposition 2.6 it can be transformed into an n -compressed family $\tilde{\mathcal{G}}$ by a sequence of small modifications, without decreasing the measure. Then Corollary 2.10 can be applied to $\tilde{\mathcal{G}}$ to increase the measure again. As we show in Section 4 below, the process can be continued until the n 'th coordinate becomes non-influential, i.e., the effective number of coordinates decreases. Then one may repeat the whole process with the $(n-1)$ 'th coordinate etc., so that ultimately, \mathcal{F} can be transformed into a junta by a sequence of small modifications.

2.4 Cross-Intersecting Families

Definition 2.11. *Families $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$ are called cross-intersecting if for any $A \in \mathcal{F}$, $B \in \mathcal{G}$ we have $A \cap B \neq \emptyset$.*

The first generalization of the Erdős-Ko-Rado theorem to cross-intersecting families was obtained in 1968 by Kleitman [22], and since then, a multitude of EKR-type results for cross-intersecting families were proved. Such results assert that if \mathcal{F}, \mathcal{G} are cross-intersecting then they cannot be ‘large simultaneously’, when the latter can be expressed in various ways. We use two such results: the first was proved in [10], and the second follows easily from a result of [10] and an old theorem of Hilton.

Proposition 2.12 ([10], Lemma 2.7). *Let $0 < p \leq 1/2$, and let $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$ be cross-intersecting families. Then $\mu_p(\mathcal{F}) \leq (1 - \mu_p(\mathcal{G}))^{\log_{1-p} p}$.*

Note that equality holds in Proposition 2.12 when $\mathcal{F} = \{S \subset [n] : B \subset S\} =: \text{AND}_B$, and $\mathcal{G} = \{S \subset [n] : B \cap S \neq \emptyset\} =: \text{OR}_B$, where $B \subset [n]$.

Before we state the second result, we need a few preliminaries.

Notation 2.13. *For $X \subset \mathbb{N}$, $i \in \mathbb{N}$ and $\mathcal{F} \subset X^{(i)}$, we write $\mathcal{L}(\mathcal{F})$ for the initial segment of the lexicographic order on $X^{(i)}$ with size $|\mathcal{F}|$. We say a family $\mathcal{C} \subset X^{(i)}$ is lexicographically ordered if it is an initial segment of the lexicographic order on $X^{(i)}$, i.e., $\mathcal{L}(\mathcal{C}) = \mathcal{C}$.*

The following classical result was proved by Hilton (see [13], Theorem 1.2).

Proposition 2.14 (Hilton). *If $\mathcal{F} \subset [n]^{(k)}$, $\mathcal{G} \subset [n]^{(l)}$ are cross-intersecting, then $\mathcal{L}(\mathcal{F})$, $\mathcal{L}(\mathcal{G})$ are also cross-intersecting.*

Lemma 2.15 ([10], Lemma 5.7). *For any $\eta > 0$ and any $C \geq 0$, there exists $c_0 = c_0(\eta, C) \in \mathbb{N}$ such that the following holds. Let $n, l, k, d \in \mathbb{N} \cup \{0\}$ with $n \geq (1 + \eta)l + k + c_0$ and $l \geq k + c_0 - 1$. Suppose that $\mathcal{F} \subset [n]^{(l)}$, $\mathcal{G} \subset [n]^{(k)}$ are cross-intersecting, and that*

$$|\mathcal{F}| \leq \left| \text{OR}_{[d]} \cap [n]^{(l)} \right| = \binom{n}{l} - \binom{n-d}{l}.$$

Then

$$|\mathcal{F}| + C|\mathcal{G}| \leq \binom{n}{l} - \binom{n-d}{l} + C \binom{n-d}{k-d}.$$

Proposition 2.16. *Let $n, j, M \in \mathbb{N}$, $\zeta \in (0, 1/2)$, and let $k_1, k_2 \in (\zeta n, (\frac{1}{2} - \zeta)n)$ be such that $|k_2 - k_1| \leq j$. There exists $c = c(M, \zeta, j) \in \mathbb{N}$ such that the following holds. Let $\mathcal{F} \subset [n]^{(k_1)}$ and $\mathcal{G} \subset [n]^{(k_2)}$ be cross-intersecting families such that for some $d \in \{c, c+1, \dots, k_2\}$, we have*

$$\binom{n-d}{k_2-d} \leq |\mathcal{G}| \leq \binom{n-c}{k_2-c}. \quad (4)$$

Then $|\mathcal{F}| + M|\mathcal{G}| \leq \binom{n}{k_1} - \binom{n-d}{k_1} + M\binom{n-d}{k_2-d}$.

Proof. By Lemma 2.14, we may assume that \mathcal{F} and \mathcal{G} are lexicographically ordered. In addition, by an appropriate choice of c , we may assume throughout that $n \geq n_0$ for any $n_0 = n_0(M, \zeta, j) \in \mathbb{N}$.

Consider the families $\mathcal{F}_{[c]}^\emptyset$ and $\mathcal{G}_{[c]}^{[c]}$ (for c to be specified below), which are clearly cross-intersecting. As \mathcal{G} is lexicographically ordered, the assumption (4) implies $\mathcal{G} \subset \mathcal{S}_{[c]}^{[c]}$. Moreover, $\mathcal{G}_{[c]}^{[c]}$ is also lexicographically ordered, and hence, by (4) we have $\mathcal{S}_{\{c+1, \dots, d\}} \subset \mathcal{G}_{[c]}^{[c]}$. Since $\mathcal{F}_{[c]}^\emptyset$ cross-intersects $\mathcal{G}_{[c]}^{[c]}$, this implies $\mathcal{F}_{[c]}^\emptyset \subset \text{OR}_{\{c+1, \dots, d\}}$, and thus,

$$|\mathcal{F}_{[c]}^\emptyset| \leq \binom{n-c}{k_1} - \binom{n-d}{k_1}.$$

This allows us to apply Lemma 2.15 to the cross-intersecting families

$$\mathcal{F}_{[c]}^\emptyset \subset ([n] \setminus [c])^{(k_1)}, \quad \mathcal{G}_{[c]}^{[c]} \subset ([n] \setminus [c])^{(k_2-c)},$$

with the parameters $n' = n - c$, $l' = k_1$, $k' = k_2 - c$, $d' = d - c$, $C' = M$ and $\eta' = \zeta$, provided that $c := c_0(\zeta, M) + j$, to obtain

$$|\mathcal{F}_{[c]}^\emptyset| + M|\mathcal{G}_{[c]}^{[c]}| \leq \binom{n-c}{k_1} - \binom{n-d}{k_1} + M\binom{n-d}{k_2-d}. \quad (5)$$

(Note that we have $n' \geq (1 + \eta')l' + k' + c_0$, provided $n_0(M, \zeta, j)$ is sufficiently large.) Finally, we clearly have

$$|\mathcal{F}| \leq \binom{n}{k_1} - \binom{n-c}{k_1} + |\mathcal{F}_{[c]}^\emptyset|,$$

and the assumption (4) implies $|\mathcal{G}_{[c]}^{[c]}| = |\mathcal{G}|$. Therefore, (5) yields

$$\begin{aligned} |\mathcal{F}| + M|\mathcal{G}| &\leq \binom{n}{k_1} - \binom{n-c}{k_1} + |\mathcal{F}_{[c]}^\emptyset| + M|\mathcal{G}_{[c]}^{[c]}| \\ &\leq \binom{n}{k_1} - \binom{n-c}{k_1} + \binom{n-c}{k_1} - \binom{n-d}{k_1} + M\binom{n-d}{k_2-d} \\ &= \binom{n}{k_1} - \binom{n-d}{k_1} + M\binom{n-d}{k_2-d}, \end{aligned}$$

as asserted. \square

Proposition 2.16 is tight for the (uniform) ‘OR’ and ‘AND’ families, i.e. when $\mathcal{F} = \{S \in [n]^{(k_1)} : S \cap D \neq \emptyset\}$ and $\mathcal{G} = \{S \in [n]^{(k_2)} : D \subset S\}$, where $D \subset [n]$.

2.5 Reduction from k -element Sets to the Biased Measure Setting

As shown in several previous works (e.g., [8, 16]), EKR-type results for (t) -intersecting subsets of $[n]^{(k)}$, for a sufficiently large n , can be proved by reduction to similar results on the μ_p measure of (t) -intersecting subsets of $\mathcal{P}([n])$, for an appropriately chosen p . In this subsection we present the lemmas required for performing such a reduction for the stability version of the Ahlswede-Khachatrian theorem.

The reduction (in our case) works as follows. Let $\mathcal{F} \subset [n]^{(k)}$ be a t -intersecting family with $|\mathcal{F}| > f(n, k, t) - \epsilon \binom{n}{k}$. We let \mathcal{F}^\uparrow be the increasing family generated by \mathcal{F} (i.e., the minimal increasing family that contains \mathcal{F}). We take p slightly larger than $\frac{k}{n}$, and show that:

$$(a) \mu_p(\mathcal{F}^\uparrow) \gtrsim \frac{|\mathcal{F}|}{\binom{n}{k}} > \frac{f(n, k, t)}{\binom{n}{k}} - \epsilon,$$

$$(b) f(n, p, t) \sim \frac{f(n, k, t)}{\binom{n}{k}},$$

$$(c) \mu_p(\mathcal{F}^\uparrow \setminus \mathbb{A}\mathbb{K}_{t, \zeta}^n) \sim \frac{|\mathcal{F} \setminus (\mathbb{A}\mathbb{K}_{t, \zeta}^n)^{(k)}|}{\binom{n}{k}}.$$

This essentially reduces stability for subsets of $[n]^{(k)}$ to stability in the μ_p setting. We present now three propositions that justify the ‘ \sim ’ in (a)-(c). These propositions, or close variants thereof, were proved in previous works; as they do not appear in the exact form we use them, we present the simple proofs here.

The first proposition, which was essentially proved by Friedgut [16], shows that (a) holds.

Proposition 2.17 (Friedgut). *Let $\zeta, \delta \in (0, 1)$, and $p \in [\zeta, 1 - \zeta]$. For any $k \in [\zeta n, (1 - \zeta)n]$ such that $p \geq \frac{k + \sqrt{2n \log(\frac{1}{\delta})}}{n}$ and for any increasing $\mathcal{F} \subset \mathcal{P}([n])$,*

$$\mu_p(\mathcal{F}) > \frac{|\mathcal{F}^{(k)}|}{\binom{n}{k}} (1 - \delta).$$

Proposition 2.17 follows immediately from a simple corollary of the classical Kruskal-Katona theorem [19, 23], known as the ‘weak Kruskal-Katona theorem’.

Proposition 2.18 (Weak Kruskal-Katona Theorem). *Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family. For any $1 \leq k \leq m \leq n$, we have $|\mathcal{F}^{(m)}|/\binom{n}{m} \geq |\mathcal{F}^{(k)}|/\binom{n}{k}$.*

Proof of Proposition 2.17. By Proposition 2.18, we have

$$\begin{aligned} \mu_p(\mathcal{F}) &\geq \sum_{m=k}^n \frac{|\mathcal{F}^{(m)}|}{\binom{n}{m}} \mu_p([n]^{(m)}) \geq \frac{|\mathcal{F}^{(k)}|}{\binom{n}{k}} \sum_{m=k}^n \mu_p([n]^{(m)}) = \frac{|\mathcal{F}^{(k)}|}{\binom{n}{k}} \mu_p(\{S \subset [n] : |S| \geq k\}) \\ &\geq \frac{|\mathcal{F}^{(k)}|}{\binom{n}{k}} (1 - \delta), \end{aligned}$$

where the last inequality follows from the choice of p by a standard Chernoff bound. \square

The second proposition, proved by Dinur and Safra [9], shows that (b) holds.

Proposition 2.19 (Dinur and Safra). *Let $j \in \mathbb{N}$, $\zeta, \epsilon \in (0, 1)$, and $p \in [\zeta, 1 - \zeta]$. There exist $\delta' = \delta'(\epsilon, j)$ and $n_0 = n_0(j, \zeta, \epsilon) \in \mathbb{N}$ such that the following holds for all $n > n_0$. For any $k \in [\zeta n, (1 - \zeta)n] \cap \mathbb{N}$ such that $|p - \frac{k}{n}| < \delta'$ and for any j -junta $\mathcal{J} \subset \mathcal{P}([n])$, we have*

$$\left| \mu_p(\mathcal{J}) - \frac{|\mathcal{J} \cap [n]^{(k)}|}{\binom{n}{k}} \right| < \epsilon. \quad (6)$$

Proof. Assume w.l.o.g. that \mathcal{J} depends only on the coordinates in $[j]$. Since j is fixed, it is sufficient to prove that (6) holds when $\mathcal{J} = \mathcal{J}_C := \{A \subset [n] : A \cap [j] = C\}$ for any $C \subset [j]$. And indeed,

$$\begin{aligned} |\mu_p(\mathcal{J}) - \frac{|\{A \in [n]^{(k)} : A \cap [j] = C\}|}{\binom{n}{k}}| &= \left| p^{|C|} (1-p)^{j-|C|} - \frac{\binom{n-j}{k-|C|}}{\binom{n}{k}} \right| \\ &= \left| p^{|C|} (1-p)^{j-|C|} - \frac{k \cdot \dots \cdot (k-|C|+1) \cdot (n-k) \cdot \dots \cdot (n-k+1-j+|C|)}{n \cdot \dots \cdot (n-j+1)} \right| \\ &< \left| p^{|C|} (1-p)^{j-|C|} - \left(\frac{k}{n}\right)^{|C|} \left(\frac{n-k}{n}\right)^{j-|C|} \right| + o_{n \rightarrow \infty}(1) < \epsilon, \end{aligned}$$

where n_0, δ' clearly can be chosen such that the last inequality holds. \square

The third proposition, a variant of which was proved by Dinur and Friedgut [8], shows that (c) holds.

Proposition 2.20 (Dinur and Friedgut). *Let $j \in \mathbb{N}$, $\zeta, \delta'' \in (0, 1)$, and $p \in [\zeta, 1 - \zeta]$. There exist $C = C(\zeta, j) > 0$ and $n_0(t, \zeta, \delta'') \in \mathbb{N}$ such that the following holds for all $n > n_0$ and all $k \in [\zeta n, (1 - \zeta)n] \cap \mathbb{N}$ such that $p > \frac{k + \sqrt{2n \log 2}}{n}$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family, and let \mathcal{J} be a j -junta such that $\mu_p(\mathcal{F} \setminus \mathcal{J}) < \delta''$. Then*

$$|(\mathcal{F} \setminus \mathcal{J})^{(k)}| < C \delta'' \binom{n}{k}.$$

Proof. Suppose w.l.o.g. that \mathcal{J} depends on the coordinates in $[j]$. Since j is fixed, it is sufficient to prove that for any $E \notin \mathcal{J}$, we have

$$|\{A \in \mathcal{F}^{(k)} : A \cap [j] = E\}| \leq C' \delta'' \binom{n}{k},$$

for some $C' = C'(\zeta, j) > 0$. We show that

$$|\{A \in \mathcal{F}^{(k)} : A \cap [j] = E\}| < \frac{2\delta''}{p^{|E|} (1-p)^{j-|E|}} \binom{n-j}{k-|E|}, \quad (7)$$

which is sufficient, as the right hand side of (7) is $\leq C' \delta'' \binom{n}{k}$ by the proof of Proposition 2.19. Suppose for a contradiction that (7) fails. By Proposition 2.17 (with $\delta' = 1/2$), we have

$$\mu_p(\mathcal{F}_{[j]}^E) > \frac{2\delta''}{p^{|E|} (1-p)^{j-|E|}} \cdot (1 - 1/2) = \frac{\delta''}{p^{|E|} (1-p)^{j-|E|}}.$$

Hence, $\mu_p(\mathcal{F} \setminus \mathcal{J}) \geq p^{|E|} (1-p)^{j-|E|} \mu_p(\mathcal{F}_{[j]}^E) > \delta''$, a contradiction. This completes the proof. \square

3 A Bootstrapping Lemma

In this section we present a bootstrapping argument showing that if a t -intersecting \mathcal{F} is already ‘somewhat’ close to $\mathbb{A}\mathbb{K}_{t,\zeta}^n$, then it must be ‘very’ close to $\mathbb{A}\mathbb{K}_{t,\zeta}^n$. We use this argument to show that there exists a ‘barrier’ in the distance of \mathcal{F} from $\mathbb{A}\mathbb{K}_{t,\zeta}^n$ that cannot be crossed by performing only small modifications.

Lemma 3.1 (Bootstrapping Lemma). *Let $t \in \mathbb{N}$ and let $\zeta > 0$. Then there exists $C = C(t, \zeta) > 0$ such that the following holds. Let $\zeta \leq p \leq 1/2 - \zeta$, let $\epsilon > 0$, let $\mathcal{F} \subset \mathcal{P}([n])$ be a t -intersecting family, and let $\mathcal{G} \in \mathbb{AK}_{t, \zeta}^n$. If*

$$\mu_p(\mathcal{F} \cap \mathcal{G}) \geq \mu_p(\mathcal{G})(1 - \epsilon),$$

then

$$\mu_p(\mathcal{F} \setminus \mathcal{G}) \leq C\epsilon^{\log_{1-p} p}.$$

Proof. Without loss of generality, we may assume that $\mathcal{G} = \mathcal{F}_{n, t, r}$ for some $r \in \mathbb{N}$. Note that, since $p \leq 1/2 - \zeta$, we have $r \leq r_0(t, \zeta)$ (this argument was already used in the proof of Proposition 2.3). Hence, the assumption $\mu_p(\mathcal{F} \cap \mathcal{F}_{n, t, r}) \geq \mu_p(\mathcal{F}_{n, t, r})(1 - \epsilon)$ implies that for any $D \in \mathcal{F}_{2r+t, t, r}$, we have

$$\mu_p(\mathcal{F}_{[2r+t]}^D) \geq 1 - O_{t, \zeta}(\epsilon). \quad (8)$$

It is clear that for each $E \notin \mathcal{F}_{2r+t, t, r}$, there exists $D \in \mathcal{F}_{2r+t, t, r}$ such that $|D \cap E| \leq t - 1$. Since \mathcal{F} is t -intersecting, for any such D, E , the families $\mathcal{F}_{[2r+t]}^D$ and $\mathcal{F}_{[2r+t]}^E$ are cross-intersecting. By Proposition 2.12 and (8), this implies

$$\mu_p(\mathcal{F}_{[2r+t]}^E) \leq \left(1 - \mu_p(\mathcal{F}_{[2r+t]}^D)\right)^{\log_{1-p} p} \leq (O_{t, \zeta}(\epsilon))^{\log_{1-p} p} = O_{t, \zeta}(\epsilon^{\log_{1-p} p}).$$

Therefore,

$$\begin{aligned} \mu_p(\mathcal{F} \setminus \mathcal{F}_{n, t, r}) &= \sum_{E \notin \mathcal{F}_{2r+t, t, r}} p^{|E|} (1-p)^{2r+t-|E|} \mu_p(\mathcal{F}_{[2r+t]}^E) \\ &\leq O_{t, \zeta}(\epsilon^{\log_{1-p} p}) \sum_{E \in \mathcal{P}[2r+t]} p^{|E|} (1-p)^{2r+t-|E|} = O_{t, \zeta}(\epsilon^{\log_{1-p} p}). \end{aligned}$$

This completes the proof. \square

The following corollary shows that in order to prove Theorem 1.5, it is sufficient to show that as $\mu_p(\mathcal{F}) \rightarrow f(n, p, t)$, the distance $\mu_p(\mathcal{F} \setminus \mathbb{AK}_{t, \zeta}^n)$ is smaller than a sufficiently small constant c .

Corollary 3.2. *Let $t \in \mathbb{N}$ and $\zeta > 0$. There exist positive constants \tilde{C}, c, ϵ_0 depending only on ζ and t , such that for any t -intersecting family $\mathcal{F} \subset \mathcal{P}([n])$ and any $p \in [\zeta, 1/2 - \zeta]$, if*

$$\mu_p(\mathcal{F} \setminus \mathbb{AK}_{t, \zeta}^n) \leq c \quad \text{and} \quad \mu_p(\mathcal{F}) \geq f(n, p, t)(1 - \epsilon)$$

for some $\epsilon \leq \epsilon_0$, then

$$\mu_p(\mathcal{F} \setminus \mathbb{AK}_{t, \zeta}^n) \leq \tilde{C}\epsilon^{\log_{1-p} p}.$$

Proof. By the assumption on \mathcal{F} , there exists $\mathcal{G} \in \mathbb{AK}_{t, \zeta}^n$ such that

$$\mu_p(\mathcal{F} \cap \mathcal{G}) \geq f(n, p, t)(1 - \epsilon) - c.$$

Hence, by Lemma 3.1, we have $\mu_p(\mathcal{F} \setminus \mathbb{AK}_{t, \zeta}^n) \leq C(\epsilon + c/f(n, p, t))^{\log_{1-p} p}$ for some $C = C(t, \zeta)$. Let c be sufficiently small (as a function of t, ζ) such that $C \cdot (2c/f(n, p, t))^{\log_{1-p} p} \leq c/2$ for all $p \in [\zeta, 1/2 - \zeta]$. If $\epsilon > c/f(n, p, t)$, then

$$\mu_p(\mathcal{F} \setminus \mathbb{AK}_{t, \zeta}^n) \leq C(\epsilon + c/f(n, p, t))^{\log_{1-p} p} \leq C \cdot (2\epsilon)^{\log_{1-p} p} \leq \tilde{C}\epsilon^{\log_{1-p} p},$$

and we are done. Otherwise, we have

$$\mu_p(\mathcal{F} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) \leq C(\epsilon + c/f(n,p,t))^{\log_{1-p} p} \leq C \cdot (2c/f(n,p,t))^{\log_{1-p} p} < c/2.$$

In such a case, we can repeat the process with the same ϵ and $c/2$ instead of c . At some stage c will become sufficiently small so that $\epsilon > c/f(n,p,t)$, and then (as in the first case) we have $\mu_p(\mathcal{F} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) \leq \tilde{C}\epsilon^{\log_{1-p} p}$, as asserted. \square

Finally, we can use the proof of Corollary 3.2 to show the existence of a barrier that cannot be crossed by small modifications.

Corollary 3.3. *Let $t \in \mathbb{N}$ and let $\zeta > 0$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be t -intersecting, and let $p \in [\zeta, 1/2 - \zeta]$. Let c, ϵ_0 be as in Corollary 3.2, and let $\epsilon_1 := \min(\epsilon_0, c/f(n,p,t))$. Suppose that*

$$\mu_p(\mathcal{F}) \geq f(n,p,t)(1 - \epsilon_1) \quad \text{and} \quad \mu_p(\mathcal{F} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) > c.$$

Let $\mathcal{G} \subset \mathcal{P}([n])$ be a t -intersecting family with $\mu_p(\mathcal{G}) > \mu_p(\mathcal{F})$ which is a $(c/2)$ -small modification of \mathcal{F} (i.e., $\mu_p(\mathcal{F} \setminus \mathcal{G}) < \frac{c}{2}$). Then

$$\mu_p(\mathcal{G} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) > c.$$

Proof. Suppose for a contradiction that $\mu_p(\mathcal{G} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) \leq c$. By the proof of Corollary 3.2, we have $\mu_p(\mathcal{G} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) < \frac{c}{2}$. This yields

$$\mu_p(\mathcal{F} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) \leq \mu_p(\mathcal{F} \setminus \mathcal{G}) + \mu_p(\mathcal{G} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) < c,$$

a contradiction. \square

4 Proof of Theorem 1.5

Let us recall the formulation of Theorem 1.5.

Theorem. *For any $t \in \mathbb{N}$ and any $\zeta > 0$, there exists $C = C(t, \zeta) > 0$ such that the following holds. Let $p \in [\zeta, \frac{1}{2} - \zeta]$, and let $\epsilon > 0$. If $\mathcal{F} \subset \mathcal{P}([n])$ is a t -intersecting family such that $\mu_p(\mathcal{F}) \geq f(n,p,t)(1 - \epsilon)$, then there exists a family \mathcal{G} isomorphic to some $\mathcal{F}_{n,t,r}$, such that $\mu_p(\mathcal{F} \setminus \mathcal{G}) \leq C\epsilon^{\log_{1-p} p}$.*

Proof. Let c, ϵ_1 be as in Corollary 3.3, and let $\epsilon_2 := \epsilon \left(t, \zeta, \max \left(t + 2\ell(t, \zeta), t/(2\zeta), \frac{4}{c^2 p(1-p)} \right) \right)$ in the notations of Proposition 2.3. Define $\epsilon_3 := \min(\epsilon_1, \epsilon_2)$. Let $r := \ell(t, \zeta)$.

Let $\mathcal{F} \subset \mathcal{P}([n])$ be a t -intersecting family. By replacing \mathcal{F} with \mathcal{F}^\uparrow , we may assume that \mathcal{F} is increasing. We may also assume that $\mu_p(\mathcal{F}) \geq f(n,p,t)(1 - \epsilon_3)$. (There is no loss of generality in this assumption, as C can be chosen such that the theorem holds trivially for all $\epsilon > \epsilon_3$.) We would like to show that $\mu_p(\mathcal{F} \setminus \mathbb{A}\mathbb{K}_{t,\zeta}^n) \leq c$. This will complete the proof of the theorem by Corollary 3.2.

We let $\mathcal{F}_0 = \mathcal{F}$ and construct a sequence (\mathcal{F}_i) of increasing t -intersecting families such that each \mathcal{F}_i is obtained from \mathcal{F}_{i-1} by a series of $c/2$ -small modifications. Each ‘step’ in the sequence is composed of compression (using the process of Section 2.3.1) and measure increase (using the process of Section 2.3.2). The construction of \mathcal{F}_i from \mathcal{F}_{i-1} is defined as follows:

1. If either \mathcal{F}_{i-1} depends on at most $\max\{t + 2r, t/(2\zeta)\}$ coordinates, or else

$$\min_{j: I_j(\mathcal{F}_{i-1}) > 0} I_j(\mathcal{F}_{i-1}) \geq c/2,$$

then stop.

2. Consider the set of coordinates with non-zero influence on \mathcal{F}_{i-1} . Assume w.l.o.g. that this set is $[m]$, and that $\min_{j \in [m]} I_j(\mathcal{F}_{i-1}) = I_m(\mathcal{F}_{i-1})$. Transform \mathcal{F}_{i-1} to an m -compressed increasing t -intersecting family \mathcal{G}_{i-1} with $\mu_p(\mathcal{G}_{i-1}) = \mu_p(\mathcal{F}_{i-1})$ by a sequence of small modifications (as described in Proposition 2.6).
3. Transform \mathcal{G}_{i-1} into an increasing t -intersecting family \mathcal{F}_i with $\mu_p(\mathcal{F}_i) > \mu_p(\mathcal{G}_{i-1})$ by a small modification (as described in Corollary 2.10, which can be applied since $m > t/(2\zeta)$) and then taking the up-closure to turn the family into an increasing family.

We claim that during all the process, all modifications are $c/2$ -small, and that the process terminates after a finite number of steps.

Indeed, Proposition 2.6 assures that all modifications in the m -compression process are $I_m(\mathcal{F}_{i-1})$ -small, and we have $I_m(\mathcal{F}_{i-1}) < c/2$, as otherwise the process terminates by (1.). Similarly, Corollary 2.10 (which can be applied to \mathcal{G}_{i-1} , since \mathcal{G}_{i-1} is m -compressed) assures that the transformation to \mathcal{F}_i is an $I_m(\mathcal{G}_{i-1})$ -small modification, and by Proposition 2.6, $I_m(\mathcal{G}_{i-1}) \leq I_m(\mathcal{F}_{i-1}) < c/2$. By the construction, the sequence of measures $(\mu_p(\mathcal{F}_i))$ is strictly monotone increasing in i . As the measure of a family $\mathcal{F}_i \subset \mathcal{P}([n])$ can assume only a finite number of values, the sequence eventually terminates.

Let \mathcal{F}_ℓ be the last element of the sequence. As the sequence terminated at the ℓ 'th step, either \mathcal{F}_ℓ depends on at most $\max\{t + 2r, t/(2\zeta)\}$ coordinates or else $\min_{j: I_j(\mathcal{F}_\ell) \neq 0} I_j(\mathcal{F}_\ell) \geq c/2$. In the latter case, by Proposition 2.2 \mathcal{F}_ℓ depends on at most $\frac{4}{c^2 p(1-p)}$ coordinates. Thus, in either

case \mathcal{F}_ℓ depends on at most $\max\left\{t + 2r, t/(2\zeta), \frac{4}{c^2 p(1-p)}\right\}$ coordinates. Since $\mu_p(\mathcal{F}_\ell) > \mu_p(\mathcal{F}) \geq f(n, p, t)(1 - \epsilon_3)$, Proposition 2.3 implies that $\mathcal{F}_\ell \in \mathbb{AK}_{t, \zeta}^n$. In particular, $\mu_p(\mathcal{F}_\ell \setminus \mathbb{AK}_{t, \zeta}^n) = 0 < c$.

Now, we unroll the steps of the sequence. As \mathcal{F}_ℓ was obtained from $\mathcal{F}_{\ell-1}$ by a $c/2$ -small modification, Corollary 3.3 implies $\mu_p(\mathcal{F}_{\ell-1} \setminus \mathbb{AK}_{t, \zeta}^n) < c$. The same holds for any step of the sequence, and thus, by (reverse) induction, we get $\mu_p(\mathcal{F} \setminus \mathbb{AK}_{t, \zeta}^n) = \mu_p(\mathcal{F}_0 \setminus \mathbb{AK}_{t, \zeta}^n) < c$. As mentioned above, this completes the proof of the theorem by Corollary 3.2. \square

4.1 Tightness of Theorem 1.5

As mentioned in the introduction, Theorem 1.5 is tight (up to a factor depending upon t and ζ alone) for the families $\mathcal{G}_{t, r, s}$, defined as:

$$\begin{aligned} \mathcal{G}_{t, r, s} = \{ & A \subset \mathcal{P}([n]) : |A \cap [t + 2r]| \geq t + r, A \cap \{t + 2r + 1, \dots, t + 2r + s\} \neq \emptyset \\ & \cup \{A \subset \mathcal{P}([n]) : |A \cap [t + 2r]| = t + r - 1, \{t + 2r + 1, \dots, t + 2r + s\} \subset A\} \}, \end{aligned}$$

for all sufficiently large s . Here is the computation showing this. Let $\zeta \leq p \leq 1/2 - \zeta$. Choose $r \in \mathbb{N} \cup \{0\}$ such that $p \in \left[\frac{r}{t+2r-1}, \frac{r+1}{t+2r+1}\right]$, so that $\mu_p(\mathcal{F}_{n, t, r}) = f(n, p, t)$. We have

$$\mu_p(\mathcal{G}_{t, r, s}) = f(n, p, t)(1 - (1 - p)^s) + \binom{t + 2r}{t + r - 1} p^{t+r-1} (1 - p)^{r+1} p^s,$$

and

$$\mu_p(\mathcal{G}_{t,r,s} \setminus \mathcal{F}_{n,t,r}) = \binom{t+2r}{t+r-1} p^{t+r-1} (1-p)^{r+1} p^s.$$

As $\zeta < p < 1/2 - \zeta$, we have $r = O_{t,\zeta}(1)$ and therefore $\binom{t+2r}{t+r-1} p^{t+r-1} (1-p)^{r+1} = \Theta_{t,\zeta}(1)$. Hence, for all $s \geq s_0(t, \zeta)$, we have

$$\mu_p(\mathcal{G}_{t,r,s}) = f(n, p, t) (1 - (1-p)^s) + \Theta_{t,\zeta}(1) p^s \geq f(n, p, t) (1 - \frac{1}{2} (1-p)^s),$$

while $\mu_p(\mathcal{G}_{t,r,s} \setminus \mathcal{F}_{n,t,r}) = \Theta_{t,\zeta}(1) p^s$. Denoting $\epsilon := \frac{1}{2}(1-p)^s$, we have $\mu_p(\mathcal{G}_{t,r,s}) \geq f(n, p, t) (1 - \epsilon)$ and $\mu_p(\mathcal{G}_{t,r,s} \setminus \mathcal{F}_{n,t,r}) = \Theta_{t,\zeta}(\epsilon^{\log_{1-p} p})$, which is tight for Theorem 1.5.

5 Proof of Theorem 1.6

In this section we present the proof of Theorem 1.6. First, we deduce a ‘weak stability’ result from Theorem 1.5 by the reduction technique presented in Section 2.5. Then, we use a ‘bootstrapping’ technique similar to in the proof of Lemma 3.1, to leverage the weak stability result into the assertion of the theorem.

Proposition 5.1 (A weak stability theorem). *Let $t \in \mathbb{N}$ and $\zeta, \epsilon > 0$. There exist $C = C(t, \zeta) > 0$ and $n_0(t, \zeta, \epsilon) \in \mathbb{N}$ such that the following holds for all $n > n_0$, all $k \in [\zeta n, (1/2 - \zeta)n] \cap \mathbb{N}$ and $p = \frac{k + \sqrt{4n \log n}}{n}$. Let $\mathcal{F} \subset [n]^{(k)}$ be a t -intersecting family that satisfies $|\mathcal{F}| \geq f(n, k, t) - \epsilon \binom{n}{k}$. Then there exists \mathcal{G} isomorphic to some $\mathcal{F}_{n,k,t,r}$ such that $|\mathcal{F} \setminus \mathcal{G}| \leq C \epsilon^{\log_{1-p} p} \binom{n}{k}$.*

Proof. Let \mathcal{F}, k and p satisfy the assumption of the proposition, and let $\mathcal{F}^\uparrow \subset \mathcal{P}([n])$ be the increasing family generated by \mathcal{F} . By Proposition 2.17, for a sufficiently large n ,

$$\mu_p(\mathcal{F}^\uparrow) \geq \left(\frac{f(n, k, t)}{\binom{n}{k}} - \epsilon \right) \left(1 - \frac{1}{n} \right) \geq \frac{f(n, k, t)}{\binom{n}{k}} - 2\epsilon.$$

Let $\mathcal{F}_{n,p,t,r} \in \mathbb{AK}_{t,\zeta}^n$ be a family for which the maximal μ_p measure is attained, i.e., $\mu_p(\mathcal{F}_{n,p,t,r}) = f(n, p, t)$. As $p \leq 1/2 - \zeta/2$ (which holds assuming that n is sufficiently large), $\mathcal{F}_{n,p,t,r}$ depends on at most j coordinates, for some $j = j(t, \zeta) \in \mathbb{N}$. Hence, by Proposition 2.19,

$$f(n, p, t) = \mu_p(\mathcal{F}_{n,p,t,r}) < \frac{|\mathcal{F}_{n,p,t,r}^{(k)}|}{\binom{n}{k}} + \epsilon \leq \frac{f(n, k, t)}{\binom{n}{k}} + \epsilon.$$

Thus, $\mu_p(\mathcal{F}^\uparrow) \geq f(n, p, t) - 3\epsilon$. Since \mathcal{F}^\uparrow is t -intersecting, we can apply to it Theorem 1.5 to get

$$\mu_p(\mathcal{F}^\uparrow \setminus \tilde{\mathcal{G}}) \leq C' \epsilon^{\log_p(1-p)},$$

for some $\tilde{\mathcal{G}} \in \mathbb{AK}_{t,\zeta}^n$ and $C' = C'(t, \zeta)$. Finally, denoting $\mathcal{G} := \tilde{\mathcal{G}}^{(k)}$, we obtain by Proposition 2.20

$$|\mathcal{F} \setminus \mathcal{G}| = |(\mathcal{F}^\uparrow \setminus \tilde{\mathcal{G}})^{(k)}| < C \epsilon^{\log_{1-p} p} \binom{n}{k},$$

for a sufficiently large $C = C(t, \zeta)$, as asserted. \square

Proposition 5.1 shows that the assertion of Theorem 1.6 holds for all $n \geq n_0(t, \zeta, \epsilon)$. This is not sufficient for Theorem 1.6, in proving which we may only assume n to be large in terms of t, ζ (and not in terms of ϵ). However, we can apply Proposition 5.1 with any *moderately small* $\epsilon_0(t, \zeta) > 0$ to conclude that for any $n \geq n_1(t, \zeta)$, if \mathcal{F} satisfies the assumption of Theorem 1.6 then there exists $\mathcal{G} \cong \mathcal{F}_{n,k,t,r}$ such that $|\mathcal{F} \setminus \mathcal{G}| \leq \epsilon_0 \binom{n}{k}$. In the proof of Theorem 1.6 below, we use this weak stability version, with ϵ_0 chosen in such a way that we will be able to use Proposition 2.16 to bootstrap the ‘weak stability’ into ‘strong stability’.

Let us recall the formulation of Theorem 1.6.

Theorem. *Let $n, t, d \in \mathbb{N}$, $\zeta \in (0, 1/2)$, and $k \in (\zeta n, (\frac{1}{2} - \zeta)n)$. There exists $C = C(t, \zeta) > 0$ such the following holds. Let $\mathcal{F} \subset [n]^{(k)}$ be a t -intersecting family with $|\mathcal{F}| > f(n, k, t) - \frac{1}{C} \binom{n-d}{k}$. Then there exists \mathcal{G} isomorphic to some $\mathcal{F}_{n,k,t,r}$ such that $|\mathcal{F} \setminus \mathcal{G}| < C \binom{n-d}{k-d}$.*

Proof of Theorem 1.6. Recall that for fixed t, ζ , all elements of $\mathbb{A}\mathbb{K}_{t,\zeta}^n$ are juntas on at most $j = j(t, \zeta)$ elements. Denote $c = c(2^j, t, \zeta)$ in the notations of Proposition 2.16. Let \mathcal{F} be a family that satisfies the assumption of the theorem (with a sufficiently large $C = C(t, \zeta) > 0$ to be specified below). Clearly, we may assume that $d \leq k + 1$. By increasing C if necessary, we may assume that $d \geq d_0(t, \zeta)$ for any $d_0(t, \zeta) \in \mathbb{N}$ and that $n \geq n_0(t, \zeta)$ for any $n_0(t, \zeta) \in \mathbb{N}$.

Provided $n_0 = n_0(t, \zeta)$ is sufficiently large, we have $\binom{n-c-j}{k-c-j} = \Theta_{t,\zeta} \left(\binom{n}{k} \right)$. Hence, we can apply Proposition 5.1 to conclude that there exists $\mathcal{G} \in (\mathbb{A}\mathbb{K}_{t,\zeta}^n)^{(k)}$ such that

$$|\mathcal{F} \setminus \mathcal{G}| < \binom{n-c-j}{k-c-j}. \quad (9)$$

Suppose w.l.o.g. that \mathcal{G} depends only on the coordinates in $[j]$, and denote by \mathcal{G}' the restriction of \mathcal{G} to $\mathcal{P}([j])$ (i.e., $\mathcal{G}' = \{A \cap [j] : A \in \mathcal{G}\}$). Let $O \notin \mathcal{G}'$ be such that $|\mathcal{F}_{[j]}^O|$ is maximal. We would like to show that $|\mathcal{F}_{[j]}^O| \leq \binom{n-d}{k-d}$. This will complete the proof, as

$$|\mathcal{F} \setminus \mathcal{G}| = \sum_{S \in \mathcal{P}([j]) \setminus \mathcal{G}'} |\mathcal{F}_{[j]}^S| \leq 2^j |\mathcal{F}_{[j]}^O|.$$

Suppose for a contradiction that

$$|\mathcal{F}_{[j]}^O| > \binom{n-d}{k-d}. \quad (10)$$

It is clear that there exists $I \in \mathcal{G}'$ such that $|I \cap O| \leq t - 1$. We have

$$|\mathcal{F}| = \sum_{S \in \mathcal{P}([j])} |\mathcal{F}_{[j]}^S| \leq |\mathcal{F}_{[j]}^I| + 2^j |\mathcal{F}_{[j]}^O| + \left(|\mathcal{G}| - \binom{n-j}{k-|I|} \right) \quad (11)$$

(where the two last summands are upper bounds on $\sum_{S \in \mathcal{P}([j]) \setminus \mathcal{G}'} |\mathcal{F}_{[j]}^S|$ and $\sum_{S \in \mathcal{G}' \setminus \{I\}} |\mathcal{F}_{[j]}^S|$, respectively). Now, we note that since \mathcal{F} is t -intersecting, the families $\mathcal{F}_{[j]}^I, \mathcal{F}_{[j]}^O$ are cross-intersecting. We have

$$|\mathcal{F}_{[j]}^O| \leq \binom{n-c-j}{k-c-j} \leq \binom{n-j-c}{k-|O|-c}$$

(where the first inequality follows from (9) and the second holds provided $n_0 = n_0(t, \zeta)$ is sufficiently large), and on the other hand,

$$|\mathcal{F}_{[j]}^O| \geq \binom{n-d}{k-d} \geq \binom{n-j-d}{k-|O|-d}$$

(where the first inequality follows from (10) and the second holds trivially). Thus, we can apply Proposition 2.16 to get

$$\left| \mathcal{F}_{[j]}^I \right| + 2^j \left| \mathcal{F}_{[j]}^O \right| \leq \binom{n-j}{k-|I|} - \binom{n-j-d}{k-|I|} + 2^j \binom{n-j-d}{k-|O|-d}.$$

By (11), this implies

$$\begin{aligned} |\mathcal{F}| &\leq \binom{n-j}{k-|I|} - \binom{n-j-d}{k-|I|} + 2^j \binom{n-j-d}{k-|O|-d} + \left(|\mathcal{G}| - \binom{n-j}{k-|I|} \right) \\ &= |\mathcal{G}| - \binom{n-j-d}{k-|I|} + 2^j \binom{n-j-d}{k-|O|-d} \\ &\leq f(n, k, t) - \frac{1}{C} \binom{n-d}{k}, \end{aligned}$$

where the last inequality holds for all $d_0 \leq d \leq k+1$ and all $n \geq n_0$, provided $C = C(t, \zeta)$, $n_0 = n_0(t, \zeta)$ and $d_0 = d_0(t, \zeta)$ are all sufficiently large. This contradicts our assumption on \mathcal{F} , completing the proof. \square

Acknowledgements

We are grateful to Gil Kalai for encouraging us to work on this project, to David Ellis for numerous useful suggestions in all stages of the project, and to Yuval Filmus for suggesting a neat way to present the proof of Theorem 1.5, which we happily adopted.

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